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# Toda Theories as Contractions of Affine Toda Theories

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**Abstract**

Using a contraction procedure, we obtain Toda theories and their structures, from affine Toda theories and their corresponding structures. By structures, we mean the equation of motion, the classical Lax pair, the boundary term for half line theories, and the quantum transfer matrix. The Lax pair and the transfer matrix so obtained, depend nontrivially on the spectral parameter.

A Toda field theory is an integrable theory of $r$ scalar fields, based on a semisimple Lie algebra [1]. The equation of motion of such a theory is

$$-\Box\rho_a + \sum_b C_{ab} e^{\rho_b} = 0, \tag{1}$$

where

$$C_{ab} := \frac{2 < \alpha_a, \alpha_b >}{< \alpha_b, \alpha_b >} =: \frac{2K_{ab}}{K_{bb}} \tag{2}$$

is the Cartan matrix of the Lie algebra, $K_{ab}$ is its Killing form, and $\alpha_a$'s are the simple roots of the Lie algebra. Now the necessary and sufficient conditions for a matrix to be the Cartan matrix of a semisimple Lie algebra are

1) $C_{ab} = \frac{2 < \alpha_a, \alpha_b >}{< \alpha_b, \alpha_b >}$,
where $< \cdot, \cdot >$ denotes a nondegenerate inner product,
2) $C_{ab} \le 0, \quad a \ne b$,
3) $C_{ab}$'s are integers,
4) the set of $\alpha_a$'s are linearly independent.

It has been shown, however, that not all of these conditions are necessary for the theory to be integrable; in fact, one can omit the last condition without destroying the integrability of the theory [1, 2, 3]. If $\alpha_a$'s are linearly dependent, the corresponding theory is called an affine Toda theory.

The matrices satisfying the first three conditions have been classified [4]. It is shown that if no proper (and nonempty) subset of $\alpha_a$'s exists which is orthogonal to others, then there is just one linear combination of $\alpha_a$'s which is equal to zero:

$$\sum_a n^a \alpha_a = 0 \tag{3}$$

and all of the coefficients of this linear combination can be chosen positive. So, eliminating any $\alpha_a$ from the system, reduces it to the simple root system of an ordinary semisimple Lie algebra. Therefore, to obtain an affine Cartan matrix one must add a vector to the simple root system of an ordinary semisimple Lie algebra, in a way which preserves the properties 1–3. One way of doing this is to add the lowest root (the negative of

the highest root) to the system. The affine Cartan matrix so formed, is called untwisted and the coefficients $n^a$ are then the Kac lables. For some Lie algebras there exist other ways of extension. The affine Cartan matrices so formed are called twisted. There exists a complete classification of all of these extensions in the literature.

There are certain symmetry relations in some (affine) Dynkin diagrams. People have used these relations to construct a(n affine) Lie algebra from another and the corresponding (affine) Toda theory from another one, by the procedure called folding [1] (or, its generalisation, reduction [5]).

Faddeev has introduced another kind of transformation [6]. He starts from the sine-Gordon (untwisted affine $A_1$ Toda) theory, shifts the sine-Gordon field to infinity, and obtains (by a suitable scaling of the parameters of the theory) the Liouville ($A_1$ Toda) theory. In this way, he obtains a new quantum Lax operator for the Liouville theory, which depends nontrivially on the spectral parameter.

We extend this transformation (which we call it contraction) to all of (untwisted or twisted) affine Toda theories, and obtain quantum Lax operators, which depend nontrivially on the spectral parameters, for Toda theories. It is important to mention that there already exists a Lax operator for any Toda theory. These Lax operators depend, however, trivially on the spectral parameter; that is, the determinant of these Lax operators do not depend on the spectral parameter [6].

We also show that this contraction works for the case of Toda theories on the half line as well; that is, one can construct a Toda theory on the half line from the corresponding affine Toda theory, which is already well known.

Starting from (1) and using (3) it is easily seen that

$$\Box \sum_a n^a \rho_a = 0. \tag{4}$$

This means that, there is a linear combination of the fields $\sum_a n^a \rho_a$ which is free. One can expand the field $\rho_a$ around the equilibrium position $\hat{\rho}_a$:

$$\sum_b C_{ab} e^{\hat{\rho}_b} = 0 \tag{5}$$

or

$$\sum_b K_{ab} \frac{2e^{\hat{\rho}_b}}{< \alpha_b, \alpha_b >} = 0. \tag{6}$$

Comparing the above equation with (3), we get

$$\frac{2e^{\rho_b}}{< \alpha_b, \alpha_b >} = m^2 n^b, \tag{7}$$

which leads to

$$-\Box(\rho_a - \hat{\rho}_a) + m^2 \sum_b n^b K_{ab} e^{\rho_b - \hat{\rho}_b} = 0. \tag{8}$$

Defining $\phi_a := \rho_a - \hat{\rho}_a$, the equation of motion of $\phi_a$ is

$$-\Box\phi_a + m^2 \sum_b n^b K_{ab} e^{\phi_b} = 0. \tag{9}$$

Note that according to (3), the combination $n^a \phi_a$ is free. One can consistently set this combination equal to zero. $\phi_0$ is then not an independent field.

$$n^0 \phi_0 = -\sum_{\mu=1}^{r} n^\mu \phi_\mu. \tag{10}$$

Throughout this article, indices $\{a, b, \cdots\}$ run from 0 to $r$ and $\{\mu, \nu, \cdots\}$ from 1 to $r$. The equation of motion of $\phi := \alpha^\mu \phi_\mu$ is

$$-\Box\phi + m^2 \sum_b n^b \alpha_b e^{\phi_b} = 0, \tag{11}$$

and the equation of motion of $\phi^\mu$ is

$$-\Box\phi^\mu + m^2 n^\mu (e^{\phi_\mu} - e^{\phi_0}) = 0, \tag{12}$$

where raising and lowering of the indeces are performed by the metric $K_{\mu\nu}$ which is invertible.

Consider the equation (1). Shifting the $\rho_a$ fields

$$\begin{array}{c} \rho_a \to \rho_a + \xi_a \\ \xi_I \to \infty, \quad 0 \le I \le s-1 \qquad\qquad \xi_i = 0, s \le i \le r \end{array} \tag{13}$$

we say that $\rho_I$'s are contracted. We use capital indices for contracted fields, the fields shifted to infinity, and small indices for the remainings. Also note that the ordering of the indeces is unimportant: one can eliminate any of the fields $\phi_a$ in (10)–(12). We choose the eliminated field to be a contracted one. It is obvious that the contracted fields disappear from the second term in the equation (1):

$$-\Box\rho_a + \sum_i C_{ai} e^{\rho_i} = 0. \tag{14}$$

The above equation for the $\rho_j$'s is a Toda equation:

$$-\Box\rho_j + \sum_i C_{ji} e^{\rho_i} = 0, \tag{15}$$

where $C_{ji}$ is the Cartan matrix $C_{ab}$ in which the rows and coloumns corresponding to the contracted indices have been omitted. Eliminating some of $\alpha_a$'s makes the remainings linearly independent, and the remaining Cartan matrix is associated to a semisimple lie algebra. In Dynkin diagram, this contraction means that, the roots with contracted indices have been removed from the corresponding affine Dynkin diagram. In the original theory there were $r+1$ fields, a linear combination of them was free. If one contracts $s$ fields, $0 \le I \le s-1$, the fields $\rho_i, \quad s \le i \le r$ are Toda fields. We will show that there are $s$ linear combinations of the fields which are free. To show this, one should find vectors $p^a$ for which

$$\sum_a p^a C_{ai} = 0 \quad \text{or} \quad \sum_a p^a K_{ai} = 0. \tag{16}$$

Multiplying the equation (14) by $p^a$, one obtains

$$-\Box \sum_a p^a \rho_a = 0. \tag{17}$$

One of the solutions of (16) is

$$p^{(0)a} = n^a. \tag{18}$$

The remaining $s-1$ solutions are

$$p^{(I)a} = K^{Ia}, \tag{19}$$

where

$$\sum_\nu K^{\mu\nu} K_{\nu\rho} = \delta^\mu_\rho, \quad \text{and} \quad K^{a0} = K^{0a} = 0. \tag{20}$$

So the fields

$$\sum_a n^a \rho_a, \quad \rho^I := \sum_\mu K^{I\mu} \rho_\mu, \quad (1 \le I \le s-1) \tag{21}$$

are free. We want to use this method for obtaining boundary terms and Lax operators, classical and quantum, of the Toda theories from the known ones for the affine Toda theories [7, 8], which are expressed in terms of $\phi$. So we study this contraction procedure in the language of $\phi$ fields. For convinience, we use the primed indices for the fields and mass parameter before contraction:

$$-\Box{\phi'}_\mu + {m'}^2 \sum_b n^b K_{\mu b} e^{{\phi'}_b} = 0, \tag{22}$$

$$-\Box\phi'^{\mu}+m'^2 n^{\mu}(e^{\phi'_{\mu}}-e^{\phi'_0})=0. \tag{23}$$

We shift the fields $\phi'_b=\phi_b+\zeta_b$, but in this case $\zeta$'s are not indepndent:

$$n^b\zeta_b=0 \tag{24}$$

or

$$n^i\zeta_i+n^I\zeta_I=0. \tag{25}$$

We choose $\zeta_i=\zeta$ and $\zeta_I=\zeta'$ for all $i$'s and $I$'s. Sending $\zeta'\to-\infty$ , $\zeta\to\infty$ and $m'\to 0$ with the condition that $m'^2e^{\zeta}=m^2$ remains finite, one arrives at

$$-\Box\phi_{\mu}+m^2\sum_i n^iK_{\mu i}e^{\phi_i}=0. \tag{26}$$

For the noncontracted indices $j$

$$-\Box\phi_j+m^2\sum_i n^iK_{ji}e^{\phi_i}=0, \tag{27}$$

and for the contracted indices $I$

$$-\Box\phi^I=0. \tag{28}$$

So the fields $\phi^I$ are free.

Now, consider the affine Toda theories on a half line. These theories have a boundary term $\mathcal{B}$, which is a function of the fields but not their derivatives and represents the boundary conditions. In [7] the generic form of the integrable boundary interaction is given by

$$\mathcal{B}'=m'\sum_0^r A_ae^{\phi'_a/2} \tag{29}$$

where the coefficients $A_a$, $a=0,\cdots r$ are a set of real numbers. This term is a generalization of the results of the sine-Gordon theory. Applying the contraction to the boundary term $\mathcal{B}'$, we obtain the boundary term $\mathcal{B}$ for the Toda theories which can be obtained from the affine ones via contraction.

$$\mathcal{B}=m\sum_i A_ie^{\alpha_i.\phi/2}. \tag{30}$$

The same procedure can be repeated for the Lax operators. Classically, the field equation of affine Toda theories may be written as the zero curvature condition

$$[\partial_x+L_x,\partial_t+L_t]=0, \tag{31}$$

where the Lax pair is [8, 1, 9]

$$L_x=\frac{1}{2}\sum_{\mu}H_{\mu}\partial_t\phi'^{\mu}+\frac{m'}{2}[\sum_{\mu}e^{\phi'_{\mu}/2}(E^+_{\mu}+E^-_{\mu})+e^{\phi'_0/2}(\lambda'E^+_0+\lambda'^{-1}E^-_0)], \tag{32}$$

$$L_t=\frac{1}{2}\sum_{\mu}H_{\mu}\partial_x\phi'^{\mu}+\frac{m'}{2}[\sum_{\mu}e^{\phi'_{\mu}/2}(E^+_{\mu}-E^-_{\mu})+e^{\phi'_0/2}(\lambda'E^+_0-\lambda'^{-1}E^-_0)], \tag{33}$$

where $H_{\mu}$'s are the cartan generators, $E^{\pm}_{\mu}$'s the step operators corresponding to the simple roots, and $E^{\pm}_0$s are the step operators associated with the added root. Contracting the fields and the parameters as below

$$\phi'_0=\phi_0+\zeta',\quad \phi_{\mu}=\phi_{\mu}+\zeta \tag{34}$$

$$m'e^{\zeta/2}=m\quad m'e^{\zeta'/2}\lambda'=m\lambda, \tag{35}$$

one arrives at the following relations for the Lax pair of the Toda equation:

$$L_x=\frac{1}{2}\sum_{\mu}H_{\mu}\partial_t\phi^{\mu}+\frac{m}{2}[\sum_{\mu}e^{\phi_{\mu}/2}(E^+_{\mu}+E^-_{\mu})+e^{\phi_0/2}\lambda E^+_0], \tag{36}$$

$$L_t = \frac{1}{2}\sum_\mu H_\mu \partial_x \phi^\mu + \frac{m}{2}[\sum_\mu e^{\phi_\mu/2}(E_\mu^+ - E_\mu^-) + e^{\phi_0/2}\lambda E_0^+]. \tag{37}$$

Note that this Lax pair of the Toda equation depends nontrivially on the spectral parameters. The spectral parameter dependence of the Lax pair (32) and (33) is nontrivial: there is no rescaling of the roots by which one can eliminate $\lambda$ [8]. It can be easily shown that if one removes the parts proportional to $E_0^\pm$ from the Lax pair (32) and (33), there remains a Lax pair which gives the Toda field equation. This Lax pair, however, does not depend on the spectral parameter or, if one transports the spectral parameter to one of $E_\mu$'s, depends on the spectral parameter trivially. Faddeev has obtained a similar Lax pair, which depends nontrivially on the spectral parameter, for the Liouville theory, by contracting the Lax pair of the sine-Gordon theory. Also note that this contraction procedure works for the quantum theory as well: the transfer matrix of the affine Toda theories, which has been obtained in [8], can be used quite similarly to obtain a tranfer matrix of the Toda theories, which depends nontrivially on the spectral parameter, that is, its determinant does depend on the spectral parameter.

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